CORRIGENDUM TO "REGULARITY FOR STABLY PROJECTIONLESS, SIMPLE C^*-ALGEBRAS"

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Abstract. An error is identified and corrected in the construction of a non-$\mathcal{Z}$-stable, stably projectionless, simple, nuclear C^*-algebra carried out in a paper by the second author.

The problem

The construction in Section 4 of the second author's paper [1], used to prove [1, Theorem 4.1], contains a vital error. The construction is meant to produce a simple C^*-algebra with perforation in its Cuntz semigroup, as an inductive limit of stably projectionless subhomogeneous C^*-algebras.

The notation set out in [1] will be reused here, mostly without recalling the definitions.

The idea is to use generalized Razak building blocks $R(\mathbb{X}, k) \subseteq C(X, M_{k+1})$ (as defined in [1, Section 4.2]) as the stably projectionless building blocks of the inductive system; the connecting maps are unitary conjugates of restrictions of diagonal maps $D_{\alpha_1,\dots,\alpha_p} : C(X, M_n) \to C(Y, M_m)$ (as defined in [1, Section 4.1]).

For generalized Razak building blocks $R(\mathbb{X}, k) \subseteq C(X, M_{k+1})$ and $R(\mathbb{Y}, \ell) \subseteq C(X, M_{\ell+1})$, [1, Proposition 4.3] characterizes when a diagonal map $D_{\alpha_1,\dots,\alpha_p} : C(X, M_{k+1}) \to C(Y, M_{\ell+1}) \otimes M_m$ is unitarily conjugate to a map which sends $R(\mathbb{X}, k)$ into $R(\mathbb{Y}, \ell) \otimes M_m$. The characterization includes the equations

$$ka_0 + (k+1)a_1 = (m - s(k+1))\ell, \text{ and} \tag{1}$$

$$kb_0 + (k+1)b_1 = (m - s(k+1))(\ell+1), \tag{2}$$

where a_0, a_1, b_0, b_1, and s count certain values of the maps $\alpha_1, \dots, \alpha_p$; they additionally satisfy

$$p = a_0 + a_1 + s\ell = b_0 + b_1 + s(\ell+1). \tag{3}$$

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In [1, Remark 4.4], a specific (parametrized) solution is provided to the condition in [1, Proposition 4.3], and this solution is used in [1, Section 4.4] to construct the example.

Implicit in the definition of diagonal maps in [1, Section 4.1] is that they are unital (as maps $C(X, M_n) \to C(Y, M_m)$). In the case of [1, Proposition 4.3], this means that

$$p(k+1) = m(\ell+1). \tag{4}$$

However, the solution provided in [1, Remark 4.4] does not satisfy (4). In fact, some algebraic manipulation of the equations in [1, Proposition 4.3] shows that there are not very many solutions at all. Certainly, suppose that $m, \ell, p, s, a_0, a_1, b_0, b_1$ satisfy (1), (2), (3), and (4). Combining (3) and (4) yields

$$(b_0 + b_1 + s(\ell+1))(k+1) = m(\ell+1).$$

Subtracting (2) from this produces $b_0 = 0$. Likewise, one obtains $a_0 = m$.

Crucial to the construction in [1] is the use of both coordinate projections and flipped coordinate projections among the eigenmaps in the diagonal map $D_{\alpha_1,\dots,\alpha_p}$. As intimated in [1, Remark 4.4], there may be up to $\max\{a_0, b_1\}$ coordinate projections and $\max\{a_1, b_0\}$ flipped coordinate projections. To get perforation, the number of coordinate projections and flipped coordinate projections needs to be a very large fraction of the total number of eigenmaps. Since solutions to [1, Proposition 4.3] necessarily have $b_0 = 0$, it is actually not possible to get perforation in the Cuntz semigroup with this kind of construction.

The solution

Here we describe a correction to the construction in [1, Section 4], permitting a correct proof of [1, Theorem 4.1]. The solution is to allow slightly more general diagonal maps which include some copies of the zero representation.

Let X, Y be compact Hausdorff spaces and let $\alpha_1, \dots, \alpha_p : Y \to X$ be continuous functions. Suppose that $m, n, r \in \mathbb{N}$ satisfy $np + r = m$. Define $D_{\alpha_1,\dots,\alpha_p;r} : C(X, M_n) \to C(Y, M_m)$ by

$$\begin{aligned} D_{\alpha_1,\dots,\alpha_p;r}(f) &:= \mathrm{diag}(f\circ\alpha_1, f\circ\alpha_2, \dots, f\circ\alpha_p, 0_r) \\ &:= \begin{pmatrix} f\circ\alpha_1 & 0 & \cdots & 0 \\ 0 & f\circ\alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & f\circ\alpha_p & 0 \\ 0 & \cdots & 0 & 0_r \end{pmatrix}, \end{aligned}$$

We have the following generalization of [1, Proposition 4.2] (the only difference being that the map $D_{\alpha_1^{(i)},\dots,\alpha_{p_i}^{(i)}}$ is replaced by the more general $D_{\alpha_1^{(i)},\dots,\alpha_{p_i}^{(i)};r_i}$). The proof is exactly the same.

Proposition 1. *Let*

$$A_1 \xrightarrow{\phi_1^2} A_2 \xrightarrow{\phi_2^3} \cdots$$

be an inductive limit, such that for each i, the algebra A_i is a subalgebra of $C(X_i, M_{m_i})$ and $\phi_i^{i+1} = Ad(u) \circ D_{\alpha_1^{(i)},\dots,\alpha_{p_i}^{(i)};r_i}$ for some unitary $u \in C(X_{i+1}, M_{m_{i+1}})$ (so that $m_{i+1} = m_i p_i + r_i$). Suppose that X_i contains a copy Y_i of $[0,1]^{d_1\cdots d_{i-1}}$ such that

- $A_i|_{Y_i} = C(Y_i, M_{m_i})$,
- *for $t = 1, \dots, d_i$, $\alpha_t^{(i)}|_{Y_{i+1}}$ takes Y_{i+1} to Y_i via the t^{th} coordinate projection $([0,1]^{d_1\cdots d_{i-1}})^{d_i} \to [0,1]^{d_1\cdots d_{i-1}}$, and*
- *for $t = d_i + 1, \dots, p_i$, $\alpha_t^{(i)}|_{Y_{i+1}} : Y_{i+1} \to X_i$ factors through the interval.*

If

$$\prod_{i=1}^{\infty} \frac{d_{i+1}}{p_i} > 0$$

and $p_i > 1$ for all i then for any $n \in \mathbb{N}$, there exists $[a], [b] \in \mathcal{C}u(\varinjlim A_i)$ and $k \in \mathbb{N}$ such that

$$(k+1)[a] \leq k[b]$$

yet $[a] \not\leq n[b]$.

We have the following generalization of [1, Proposition 4.3]; the diagonal map $D_{\alpha_1,\dots,\alpha_p}$ of [1, Proposition 4.3] is replaced by the more general $D_{\alpha_1,\dots,\alpha_p;r}$. This results in a looser condition in (ii) (compare (1), (2) to (6), (7) respectively). The proof is nearly the same and contains no new tricks.

Proposition 2. *Let $\mathbb{X} = (X, x_0, x_1), \mathbb{Y} = (Y, y_0, y_1)$ be double-pointed spaces and let k, ℓ, m, p, r be natural numbers such that*

$$p(k+1) + r = m(\ell+1). \tag{5}$$

Let $\alpha_1, \dots, \alpha_p : Y \to X$ be continuous maps. Then the following are equivalent:

(i) *There exists a unitary $u \in C(Y, M_{\ell+1}) \otimes M_m$ such that*

$$uD_{\alpha_1,\dots,\alpha_p;r}(R(\mathbb{X}, k))u^* \subseteq R(\mathbb{Y}, \ell) \otimes M_m; \text{ and}$$

(ii) *Counting multiplicity we have*

$\{\alpha_1(y_0), \dots, \alpha_p(y_0)\} = a_0\{x_0\} \cup a_1\{x_1\} \cup \ell\{z_1\} \cup \cdots \cup \ell\{z_s\}$ *and*
$\{\alpha_1(y_1), \dots, \alpha_p(y_1)\} = b_0\{x_0\} \cup b_1\{x_1\} \cup (\ell+1)\{z_1\} \cup \cdots \cup (\ell+1)\{z_s\}$

for some points $z_1, \ldots, z_s \in X$*, and some natural numbers* a_0, a_1, b_0, b_1 *satisfying*

$$ka_0 + (k+1)a_1 = (m - s(k+1) - q)\ell, \text{ and} \tag{6}$$

$$kb_0 + (k+1)b_1 = (m - s(k+1) - q)(\ell + 1), \tag{7}$$

for some $q \in \mathbb{N}$*.*

Here is a solution to (3), (5), (6), and (7), parametrized by $s, k, u \in \mathbb{N}_{>0}$; it is almost the same as the solution in [1, Remark 4.4] with the notable difference of being correct.

$$\begin{aligned}
\ell &:= k + 1 + 2u, \\
m &:= (k^2 + 3k + 1)s, \\
a_0 &:= (k+1)(k+1+u)s, \quad a_1 := ksu, \\
b_0 &:= (k+1)su, \quad b_1 := k(k+2+u)s, \\
r &:= (k^2 + 2k + ku - u)s, \\
q &:= ks, \\
p &:= (k^2 + 2ku + 3k + 3u + 2)s.
\end{aligned}$$

The construction in [1, Section 4.4] proceeds using this solution in place of the one in [1, Remark 4.4]. In essence, the only difference is that the assignment

$$m_{i+1} := m_i(k_i + 1)^2 s_i$$

is replaced by

$$m_{i+1} := m_i(k_i^2 + 3k_i + 1)s_i.$$

As opposed to the original (though incorrect) construction in [1], it is not obvious that the algebra A constructed with these corrections has a tracial state (as opposed to only having a densely defined trace). One need not be concerned that this causes problems in proving the desired properties of this example, since nowhere in the statement or proof of [1, Theorem 4.1] (nor elsewhere in [1]) is it used that A has a tracial state.

This correction thereby provides a proof of [1, Theroem 4.1].

References

[1] Aaron Tikuisis. Regularity for stably projectionless, simple C^*-algebras. *J. Funct. Anal.*, 263(5):1382–1407, 2012.

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